ISRAEL JOURNAL OF MATHEMATICS 109 (1999), 41–52

A CARDINAL DEFINED BY A POLARIZED PARTITION RELATION

BY

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ABSTRACT

We study the first cardinal κ satisfying a partition relation defined on the set of finite sequences of smaller ordinals. We show that the fact that this cardinal is $\aleph_\omega$ is equiconsistent with the existence of a measurable cardinal. Under GCH, this cardinal must be inaccessible if it has uncountable cofinality. It is shown that the GCH assumption is necessary here.

1. Introduction

We consider a partition property of the set of finite sequences below a cardinal expressed by the partition symbol

$$\kappa \rightarrow \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \end{pmatrix}^{<\omega},$$

Received May 4, 1997

which means that for every F: $[\kappa]^{<\omega} \to 2$ there is a sequence $H_0, H_1, \ldots$ of subsets of κ with $|H_i| = \alpha_i$ such that F is constant on $\Pi_{i=0}^{n} H_i$ for all $n \in \omega$. This polarized partition relation holds for any κ such that $\kappa \to (\omega)^{<\omega}$; and it is easy to verify that it is not satisfied by ω (for example, take f: $\omega^{<\omega} \to 2$ defined by $f(a_0, \ldots, a_n) = 0$ iff $a_0 = n$).

It is then natural to ask if the first cardinal satisfying the polarized partition property is a large cardinal in the classical sense. The study of this question was initiated in [CDPM].

We will use κ^* to denote the least cardinal such that

$$\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}.$$

It was shown in [CDPM] that κ^* is a limit cardinal, and that if GCH holds, it is strongly inaccessible unless it has cofinality ω. It also follows from results of [CDPM] that $\kappa_0 > \aleph_n$ for every $n \in \omega$ and that $2^\omega < \kappa^*$ (see Lemma 2.2). More generally,

THEOREM 1.1: $\aleph_{\alpha+n} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}^{\leq n}_{\aleph_\alpha}$, *but* $\aleph_{\alpha+n} \not\to \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}^{\leq n+1}_{\aleph_\alpha}$, *for every ordinal α and every $n \in \omega$.*

In other words, for every f: $[\aleph_{\alpha+n}]^{\leq n} \to \aleph_\alpha$, there exists a sequence $H_0, H_1, \ldots, H_{n-1}$ of pairs of elements of $\aleph_{\alpha+n}$ such that f is constant on $\Pi_{i=0}^{k} H_i$ for all $k < n$, but there is g: $[\aleph_{\alpha+n}]^{\leq n+1} \to \aleph_\alpha$ which is not constant on any product of $n+1$ sets each with at least 2 elements.

For proofs of this result see, for example, [EGH] and [CDP], pages 464–465.

The first candidate to satisfy the polarized partition relation is therefore $\aleph_\omega$. In this note we show that

$$\aleph_\omega \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$$

is equiconsistent with the existence of a measurable cardinal. The proof is based on results of Koepke [K] concerning the free subset problem. Along the way we obtain a model in which $\kappa^* = \aleph_\omega$ and $\aleph_\omega < 2^{\aleph_1}$. We thank Menachem Magidor for pointing out the connection with the free subset problem.

It is also shown here that it is possible for κ^* to have uncountable cofinality without being strongly inaccessible.

From this we conclude that, regardless of what the cofinality of κ^* is, this cardinal can be below 2^δ for any infinite cardinal δ except for $\delta = \aleph_0$.

2. Measurable cardinals from the polarized partition property for $\aleph_\omega$

Let S be a structure and X a subset of the universe of S. The substructure of S generated by X is denoted by $S[X]$.

A subset X of S is said to be free in S if $x \notin S[X - \{x\}]$ for all $x \in X$.

For cardinals κ, λ and μ, $\mathrm{Fr}_\mu(\kappa, \lambda)$ expresses: for every structure S with $\kappa \subseteq S$ and at most μ functions and relations, there is a subset $X \subseteq \kappa$ free in S of cardinality λ.

THEOREM 2.1 ([K]): *$\mathrm{Fr}_\omega(\aleph_\omega, \omega)$ implies that $0^\#$ exists; in fact, it implies that there is an inner model with a measurable cardinal.*

The following is from [CDPM].

LEMMA 2.2: $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$ *implies* $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}_{2^\omega}$.

In fact, for κ^* the partition relation holds with subindex λ^ω for any $\lambda < \kappa^*$. Moreover,

$$\kappa^* \to \begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix}^{<\omega}_{\lambda^\omega} \quad \text{for any } \lambda < \kappa^*.$$

LEMMA 2.3: $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}_{2^\omega}$ *implies that for every structure S with $\kappa \subseteq S$ and at most ω functions and relations there is an infinite sequence of pairs $H_0, H_1, \ldots$ of ordinals below κ such that for every formula $\phi(x_0, \ldots, x_n)$ with n free variables, $S \models \phi(a_0, \ldots, a_n)$ if and only if $S \models \phi(b_0, \ldots, b_n)$ for every choice of $(a_0, \ldots, a_n), (b_0, \ldots, b_n) \in \Pi_{i=0}^n H_i$. Moreover, the set $B = \{\cap H_i : i \in \omega\}$ is free in S.*

Proof: Let $\phi_0, \phi_1, \ldots$ be an enumeration of the formulas of the language of S, and define $F\colon \kappa^{<\omega} \to 2^\omega$ by

$$F(a_0, a_1, \ldots, a_{n-1}) = \{i \in \omega\colon \phi_i \text{ has } n \text{ free variables and } S \models \phi(a_0, a_1, \ldots, a_{n-1})\}.$$

Let $H_0, H_1, \ldots$ be a homogeneous sequence. Then if $(a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1}) \in \Pi_{i=0}^{n-1} H_i$, they satisfy exactly the same formulas (in S).

By Proposition 2.12 of [CDPM], we can assume that the pairs $H_0, H_1, \ldots$ are strictly increasing, in other words that $\max H_i < \min H_{i+1}$. Therefore the set B is infinite. To see that the set B is free in S, let $B = \{b_0, b_1, \ldots\}$, and suppose b_j and $\{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ are such that $j \notin \{i_1, \ldots, i_k\}$ and $b_j \in S[b_{i_1}, b_{i_2}, \ldots, b_{i_k}]$. Then, $b_j = \tau(b_{i_1}, b_{i_2}, \ldots, b_{i_k})$ for some term τ, and therefore $b_j = f(\tau_1(b_{i_1}, b_{i_2}, \ldots, b_{i_k}), \ldots, \tau_m(b_{i_1}, b_{i_2}, \ldots, b_{i_k}))$ for a function f. By homogeneity of $H_0, H_1, \ldots$, if $H_j = \{b_j, a_j\}$,

$$a_j = f(\tau_1(b_{i_1}, b_{i_2}, \ldots, b_{i_k}), \ldots, \tau_m(b_{i_1}, b_{i_2}, \ldots, b_{i_k})),$$

which is impossible. ∎

Combining these we get

COROLLARY 2.4: $\aleph_\omega \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$ *implies that there is an inner model with a measurable cardinal* ($\leq \aleph_\omega$).

3. $\aleph_\omega$ can have the polarized partition property

In [Sh], Shelah used a forcing construction, starting from a model with an ω-sequence of measurable cardinals, to obtain a model in which $\mathrm{Fr}_\omega(\aleph_\omega, \omega)$ holds. Koepke [K] showed that it is enough to start from a coherent sequence of Ramsey cardinals, which in turn can be obtained from a measurable cardinal by Prikry forcing. The arguments from [K] can be used as follows to show that $\aleph_\omega$ can have the polarized partition property.

The first step is to get a coherent sequence of Ramsey cardinals.

Definition 3.1: A sequence $\langle \lambda_i : i \in \omega \rangle$ of cardinals cofinal in κ is a coherent sequence of Ramsey cardinals if for every regressive $f : [\kappa]^{<\omega} \to \kappa$ (i.e. $f(x) < \min(x)$ for $x \in [\kappa]^{<\omega}$), there is a sequence of sets $< A_i : i \in \omega >$ such that:

(i) $A_i \subseteq \lambda_i$ is cofinal in λ_i, and

(ii) if $x, y \in [\kappa]^{<\omega}$, $x, y \subset \bigcup\{A_i : i \in \omega\}$ and $\mathrm{card}(x \cap A_i) = \mathrm{card}(y \cap A_i)$ for all $i \in \omega$, then $f(x) = f(y)$.

THEOREM 3.2 ([K]): *Assume κ is a measurable cardinal and $\mathbb{P}$ is the partial order for Prikry forcing. Then, in the forcing extension there is a coherent sequence of Ramsey cardinals cofinal in κ.*

In fact, the Prikry generic sequence $\kappa_0, \kappa_1, \ldots$ added by this forcing is a coherent sequence of Ramsey cardinals.

The next step is to collapse this sequence, making $\kappa_n = \omega_{2n+2}$ for each $n \in \omega$, while using the property of the sequence to obtain a partition relation. The result is also due to Koepke, but we include a proof following a line of argumentation which will be used later in different contexts.

THEOREM 3.3 ([K]): *Fix $(\kappa_i\colon i \in \omega)$ a coherent sequence of Ramsey cardinals with supremum κ. If*

$$\mathbb{P} = \{(p_i\colon i \in \omega)\colon p_0 \in \mathrm{Col}(\omega_1, \kappa_0), p_i \in \mathrm{Col}(\kappa_{i-1}^+, \kappa_i) \text{ for } 1 \leq i < \omega\},$$

then the following holds in the forcing extension of $\mathbb{P}$:

(i) $\kappa_0 = \omega_2, \kappa_1 = \omega_4, \ldots, \kappa = \omega_\omega$.

(ii) $\aleph_\omega \to \begin{pmatrix} \omega_2 \\ \omega_4 \\ \vdots \end{pmatrix}^{<\omega}$.

Proof: To simplify the construction, we will actually use the product of $\mathrm{Col}(\kappa_{i-1}^+, [\kappa_{i-1}, \kappa_i))$ for $1 \leq i < \omega$.

Let $(\kappa_i\colon i \in \omega)$ be a coherent sequence of Ramsey cardinals with supremum κ, and let $\dot{f}$ be a $\mathbb{P}$-name for a function $f\colon \kappa^{<\omega} \to 2$. Each condition $p \in \mathbb{P}$ is an ω-sequence of functions so $p \restriction [n, \infty)$ denotes the tail of this sequence starting with its n-th coordinate. For every finite sequence of ordinals $\vec{\alpha} = (\alpha_i\colon i < n)$ such that $\alpha_i \in [\kappa_{i-1}, \kappa_i)$ (here, $\kappa_{-1} = 0$), we will define a condition $p_{\vec{\alpha}}$ deciding the value of $\dot{f}(\vec{\alpha})$. We do this inductively on the length of $\vec{\alpha}$. For each $\alpha \in \kappa_0$, let p_α be a condition deciding the value of $\dot{f}(\alpha)$ and extending $\bigcup_{\xi<\alpha} p_\xi \restriction [1, \infty)$. Let $q_0 = \bigcup_{\alpha<\kappa_0} p_\alpha \restriction [1, \infty)$. Notice that q_0 is a condition in $\mathbb{P}$ (with empty first coordinate).

For each pair α, β such that $\alpha < \kappa_0 < \beta < \kappa_1$, define $p_{\alpha,\beta}$ as a condition deciding the value $\dot{f}(\alpha, \beta)$, extending p_α, q_0 and $\bigcup\{p_{\alpha',\beta'} \restriction [2, \infty)\colon \alpha' < \kappa_0, \kappa_0 \leq \beta' < \beta\}$. We also make sure that for every $\alpha', \alpha \in \kappa_0$, $p_{\alpha',\beta} \restriction [1, \infty) = p_{\alpha,\beta} \restriction [1, \infty)$. Let $q_1 = \bigcup\{p_{\alpha,\beta} \restriction [2, \infty)\colon \alpha < \kappa_0, \beta < \kappa_1\}$.

In general, for each $n \in \omega$, given a sequence $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})$, we define $p_{\vec{\alpha}}$. We do this by induction on the reverse lexicographic ordering of the n-sequences. If $\vec{\alpha}, \vec{\beta}$ are of length n, $\vec{\beta} < \vec{\alpha}$ if there is $i \leq n-1$ such that $\vec{\beta}(j) = \vec{\alpha}(j)$ for all $i < j \leq n-1$ and $\vec{\beta}(i) < \vec{\alpha}(i)$.

Let $\bar{p}_{\vec{\alpha}}$ be a condition deciding $\dot{f}(\vec{\alpha})$ and extending $p_{\vec{\alpha} \restriction n-1}$ and q_{n-1}. Also, $\forall \vec{\beta} < \vec{\alpha}$, if i is the last place they disagree, then for all $j > i$, $\bar{p}_{\vec{\beta}}(j) \subseteq \bar{p}_{\vec{\alpha}}(j)$. This can be achieved since for fixed $\vec{\alpha}$ and i, the number of such $\vec{\beta}$ is less than κ_i, and $\mathrm{Col}(\kappa_{j-1}^+, \kappa_j)$ is at least κ_i^+-closed for all $j > i$.

Now define $p_{\vec{\alpha}}$ by putting $p_{\vec{\alpha}}(0) = \bar{p}_{\vec{\alpha}}(0)$,

$$p_{\vec{\alpha}}(i) = \bigcup\{\bar{p}_{\vec{\gamma}}(i)\colon |\vec{\gamma}| = n \text{ and } \forall j \geq i(\vec{\gamma}(j) = \vec{\alpha}(j))\}.$$

Finally put $q_n = \bigcup\{p_{\vec{\alpha}} \restriction [n+1, \infty)\colon \vec{\alpha} \in \Pi_{i=0}^{n-1} \kappa_i\}$.

Notice that $p_{\vec{\alpha}} \restriction [i+1, \infty) = p_{\vec{\beta}} \restriction [i+1, \infty)$, where i is the maximal place $\vec{\alpha}$ and $\vec{\beta}$ disagree.

Now, define a regressive function $F\colon\ \kappa^{<\omega} \to \kappa$ as follows. Given $\alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_{i_0+n-1}$ in $[\kappa_{i_0-1}, \kappa_{i_0}) \times [\kappa_{i_0}, \kappa_{i_0+1}) \times \cdots \times [\kappa_{i_0+n-2}, \kappa_{i_0+n-1})$,

$$F(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}}) = \{p_{\vec{\alpha}} \restriction i_0 : \text{for all possible ways to complete } \alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_{i_0+n-1} \text{ to a sequence } \vec{\alpha} \in \Pi_{i=0}^{i_0+n-1}[\kappa_{i-1}, \kappa_i)\}.$$

If $i_0 = 0$, we have only one sequence $\vec{\alpha}$ and only one condition $p_{\vec{\alpha}}$, so in this case we put $F(\vec{\alpha}) = p_{\vec{\alpha}}(0) \restriction \alpha_0$. By fixing some reasonable well ordering of V, we identify $F(\alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_{i_0+n-1})$ with the ordinal associated to it which is indeed $< \alpha_{i_0}$ (at least when α_{i_0} is innaccessible).

Let $A_0, A_1, \ldots$ be a homogeneous sequence as given by the definition of the coherent sequence $\kappa_0, \kappa_1, \ldots$ of Ramsey cardinals.

First of all, notice that for every $\alpha \in A_0$, $p_\alpha \restriction \alpha$ takes the same value. In fact, for every $n \geq 1$, $p_{\vec{\alpha}} \restriction \min(\vec{\alpha})$ is always the same for all $\vec{\alpha} \in \Pi_{i=0}^{n-1} A_i$.

Define now a collection of conditions. For each $n \geq 1$, let $r_\emptyset^n = p_{\vec{\alpha}} \restriction \min(\vec{\alpha})$, for $\vec{\alpha} \in \Pi_{i=0}^{n-1} A_i$. Put $r_\emptyset = \bigcup_{n=0}^\infty r_\emptyset^n$.

For each $k \geq 1$ and $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{k-1}) \in \Pi_{i=0}^{k-1} A_i$ we define $r_{\vec{\alpha}}$ as follows. Put, for every $n \geq k$, $r_{\vec{\alpha}}^n = p_{\vec{\beta}} \restriction [\alpha_{k-1}, \min \vec{\beta} - \vec{\alpha})$ for any (some) $\vec{\beta} \in \Pi_{i=0}^{n-1} A_i$ end-extending $\vec{\alpha}$.

Finally, put $r_{\vec{\alpha}} = \bigcup_{n>k} r_{\vec{\alpha}}^k$.

It should be clear that for every $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{n-1}) \in \Pi_{i=0}^{n-1} A_i$,

$$p_{\vec{\alpha}} \subseteq \bigcup_{k=0}^{n-1} r_{\alpha_0, \ldots, \alpha_k}.$$

Choose now, for every $k \in \omega$, a cofinal subset H_k of A_k such that for every $\vec{\alpha}, \vec{\alpha}' \in \Pi_{i=0}^{k-1} H_i$, if $\alpha_{k-1} < \alpha'_{k-1}$, then every ordinal appearing in $r_{\vec{\alpha}}$ is below α'_{k-1}.

As a result we get that for each $n \in \omega$, the set $\{p_{\vec{\alpha}}\colon \vec{\alpha} \in \Pi_{i-0}^{n-1} H_i\}$ is pairwise compatible.

Let H_i' be a bounded initial segment of H_i such that their cardinalities tend to κ. Then the condition p obtained taking the union of all the $r_{\vec{\alpha}}$ with $\vec{\alpha}$ coming from the H_i''s, forces $(H_i: i \in \omega)$ is a homogeneous sequence for $\dot{f}$. ∎

Taking the product of this forcing with the forcing which adds κ subsets of ω_1, it is possible to obtain a generic extension in which κ^* is $\aleph_\omega$ and $2^{\aleph_1} > \aleph_\omega$. We verify first that if there is a coherent sequence of Ramsey cardinals and we force with $\mathbb{P}_1$, the poset of countable functions from κ into 2, the polarized partition relation at κ holds in the extension. It will then be clear that the two arguments can be combined to deal with the product $\mathbb{P} \times \mathbb{P}_1$, which produces a forcing extension that satisfies both $\kappa^* = \aleph_\omega$ and $2^{\aleph_1} > \aleph_\omega$.

THEOREM 3.4: *Let $(\kappa_i: i \in \omega)$ be a coherent sequence of Ramsey cardinals with supremum κ. If $\mathbb{P}_1$ is the poset of countable functions from κ into 2, then, in the forcing extension, the property* $\kappa \to \begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix}^{<\omega}$ *holds together with* $\kappa < 2^{\aleph_1}$.

Proof: The proof is similar to the previous, although it is in some sense simpler.

Given a coherent sequence of Ramsey cardinals $(\kappa_i: i \in \omega)$ with supremun κ, and $\dot{f}$, a $\mathbb{P}_1$-name for a function $f: \kappa^{<\omega} \to 2$, define for every finite sequence of ordinals $(\alpha_i: i < n)$ such that $\alpha_i \in [\kappa_{i-1}, \kappa_i)$, a condition $p_{\vec{\alpha}}$ deciding the value of $\dot{f}(\vec{\alpha})$. Satisfying that $p_{\vec{\alpha}}$ extends $p_{\vec{\alpha}} \restriction n-1$.

Given a condition p, its type is the function τ :o.t.(dom(p))$\to$ 2 given by $\tau(\xi) = p(\alpha)$, where α is the ξ-th element of dom(p) in the increasing enumeration of this set.

The function sending each $\vec{\alpha}$ to the value decided by $p_{\vec{\alpha}}$ for $\dot{f}(\vec{\alpha})$ is regressive, so there are sets A_i cofinal in κ_i (for all $i \in \omega$), and $k_i \in 2$ such that for every $n \in \omega$, $\forall \vec{\alpha} \in \Pi_{i=0}^{n-1} A_i$, $p_{\vec{\alpha}} \Vdash \dot{f}(\vec{\alpha}) = k_n$ So, we can assume that all the conditions $p_{\vec{\alpha}}$ given by sequences of the same length decide the same value for $\dot{f}(\vec{\alpha})$.

Define a regressive function F: $[\kappa]^{<\omega} \to \kappa$ by $F(\vec{\alpha})$ =type of $p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \Pi_{i-1}^{n-1} \kappa_i$. Let $(A_i: i \in \omega)$ be a homogeneous sequence for F.

Now apply again the property of the coherent sequence of Ramsey cardinals for the function G: $[\bigcup_{i \in \omega} A_i]^{<\omega} \to \kappa$ defined as follows. Given $x \in \bigcup\{A_i: i < n\}$ such that $1 \leq |x \cap A_i| \leq 2$ for all $i < n$, there are at most 2^n ways to form two n-sequences $\vec{\alpha}, \vec{\beta}$ such that $x = \vec{\alpha} \cup \vec{\beta}$. For each pair $\vec{\alpha}, \vec{\beta}$, $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ have the same countable type τ, and we can code by a countable matrix of 0's and 1's where the domains of $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ coincide and where they differ. In other words, this matrix tells us if $p_{\vec{\alpha}}(\eta) = p_{\vec{\beta}}(\nu)$ for η, ν less than the order type of their common domain. Since we have at most 2^n such matrices for each $x \in [\bigcup A_i]^n$ as above, we can define $G(x)$ as the set of these matrices taken in a prescribed order. Thus, G is a regressive function. Let $(B_i: i \in \omega)$ be homogeneous for G. Let $H_i \subseteq B_i$, H_i countable, be such that $\min B_i < \min H_i$, and such that between every two elements of H_i there is an element of B_i not in H_i.

Claim: *For each n, the set $\{p_{\vec{\alpha}}: \vec{\alpha} \in \Pi_{i=0}^{n-1} H_i\}$ consists of pairwise compatible conditions.*

To show this, suppose $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$. There must be a common member of their domains, $\xi \in \mathrm{dom}(p_{\vec{\alpha}}) \cap \mathrm{dom}(p_{\vec{\beta}})$, such that $p_{\vec{\alpha}}(\xi) \neq p_{\vec{\beta}}(\xi)$. Since $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ have the same type τ, there are η, ν in the order type of $\mathrm{dom}(p_{\vec{\alpha}})$ such that ξ is the η-th element of $\mathrm{dom}(p_{\vec{\alpha}})$ and also the ν-th element of $\mathrm{dom}(p_{\vec{\beta}})$. Let $\vec{\gamma} \in \Pi_{i=0}^{n-1} B_i$ be such that the relative positions of the elements of $\vec{\alpha}$ with respect to the elements of $\vec{\beta}$ are the same as the relative position of the elements of $\vec{\alpha}$ with respect to the elements of $\vec{\gamma}$ and the same as the relative position of the elements of $\vec{\beta}$ with respect to the elements of $\vec{\gamma}$.

This is always possible since if $\vec{\alpha}(i) < \vec{\beta}(i)$, we can put $\vec{\beta}(i) < \vec{\gamma}(i)$; if $\vec{\beta}(i) < \vec{\alpha}(i)$, we can put $\vec{\gamma}(i) < \vec{\beta}(i)$; and if $\vec{\alpha}(i) = \vec{\beta}(i)$, we put $\vec{\beta}(i) = \vec{\gamma}(i)$.

Let $x = \vec{\alpha} \cup \vec{\beta}$, $y = \vec{\alpha} \cup \vec{\gamma}$, and $z = \vec{\beta} \cup \vec{\gamma}$. By the homogeneity of $(B_i: i \in \omega)$, $G(x) = G(y) = G(z)$, therefore:

the η-th element of $\mathrm{dom}(p_{\vec{\alpha}})$ equals the ν-th element of $\mathrm{dom}(p_{\vec{\beta}})$,
the η-th element of $\mathrm{dom}(p_{\vec{\alpha}})$ equals the ν-th element of $\mathrm{dom}(p_{\vec{\gamma}})$, and
the η-th element of $\mathrm{dom}(p_{\vec{\beta}})$ equals the ν-th element of $\mathrm{dom}(p_{\vec{\gamma}})$.

Then, the ν-th element of $\mathrm{dom}(p_{\vec{\gamma}})$ must be two different ordinals at the same time, a contradiction.

If $p_n = \bigcup\{p_{\vec{\alpha}}: \vec{\alpha} \in \Pi_{i=0}^{n-1} H_i\}$, then $p_1 \geq p_2 \geq \cdots$.

The condition $p_\infty = \bigcup\{p_{\vec{\alpha}}: \vec{\alpha} \in \Pi_{i=0}^{n} H_i, n \in \omega\}$ forces that $\dot{f}$ is constant on $\Pi_{i=0}^{n} H_i$ for every $n \in \omega$. ∎

Corollary 3.5: *If there is a measurable cardinal, then there is a forcing extension in which $\kappa^* = \aleph_\omega$ and $2^{\aleph_1} > \aleph_\omega$.*

4. Uncountable cofinality

Lemma 1.6 of [CDPM] establishes that if $\mathrm{cof}(\kappa^*) > \omega$ then $2^{\mathrm{cof}(\kappa^*)} > \kappa^*$. In the presence of the GCH this implies that if $\mathrm{cof}(\kappa^*) > \omega$ then κ^* must be strongly inaccessible. In this section we show that the GCH cannot be removed. More precisely, we will show that while κ^* is always bigger than $2^{\aleph_0}$ (in view of 2.1), there are models of set theory in which κ^* is $2^{\aleph_1}$. From this, we conclude the following.

THEOREM 4.1: *If there is a cardinal κ such that $\kappa \to (\omega)^{<\omega}$, then there is a forcing extension in which the first cardinal κ such that $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$ has uncountable cofinality but is not inaccessible.*

In fact, we shall prove the following:

THEOREM 4.2: *If there is a cardinal κ such that $\kappa \to (\omega)^{<\omega}$, then there is a model in which $2^{\aleph_1}$ is the first cardinal satisfying the partition property $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.*

Proof: Suppose $\kappa \to (\omega)^{<\omega}$. We will force to add κ-many new subsets of ω_1 getting a generic extension in which $2^{\aleph_1}$ is κ and $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.

We force with the partial order $\mathbb{P}_1$ of countable partial functions from κ to 2 to add κ many Cohen subsets of $\aleph_1$. Let $<$ be a well ordering of V_κ and let $\dot{f}$ be a $\mathbb{P}_1$-name for a function from $\kappa^{<\omega}$ to 2. Since the partial order $\mathbb{P}_1$ is homogeneous, it suffices to show that not every condition of $\mathbb{P}_1$ forces that $\dot{f}$ is a counterexample to the partition relation.

Let $I \subset \kappa$ be an infinite set of indiscernibles for the structure $\langle V_\kappa, \mathbb{P}_1, <, \Vdash, \dot{f}, \xi\rangle_{\xi\in\omega_1}$.

Let $H_0, H_1, \ldots$ be a sequence of pairs of indiscernibles such that for every $i \in \omega$, there are at least two indiscernibles between $\max(H_i)$ and $\min(H_{i+1})$ and also at least two indiscernibles below $\min(H_0)$.

For each $\vec{\alpha} \in I^n$, we will define a condition $p_{\vec{\alpha}}$. This will be done inductively on the length of the sequence $\vec{\alpha}$.

For each indiscernible $\alpha \in I$, let p_α be the $<$-least condition q which decides the value of $\dot{f}(\alpha)$. Let i_1 be the value forced by p_α; thus, $p_\alpha \Vdash \dot{f}(\alpha) = i_1$. Notice that by indescernibility, i_1 is the same for all the indiscernibles. The pair $\{p_\alpha\colon \alpha \in H_0\}$ is compatible. We postpone the proof until we deal with a more general case in the inductive step.

Assume inductively that we have defined $p_{\vec{\beta}}$ for all increasing $\vec{\beta} \in I^n$, and also $i_1, \ldots, i_n$ in such a way that $p_{\vec{\beta}} \Vdash \dot{f}(\vec{\beta}) = i_n$ for all $\vec{\beta} \in I^n$. And suppose also that $\{p_{\vec{\beta}}\colon \vec{\beta} \in \Pi_{i=0}^{n-1} H_i\}$ is pairwise compatible. and that $p_{\vec{\beta}} \leq p_{\vec{\beta}\restriction n-1}$.

We define now, for every $\vec{\alpha} \in I^{n+1}$, $p_{\vec{\alpha}}$ as the $<$-least condition q such that $q \leq p_{\vec{\alpha}\restriction n-1}$ and q decides the value of $\dot{f}(\vec{\alpha})$. Call i_{n+1} the value forced by all the $p_{\vec{\alpha}}$ (by indiscernibility the same value is indeed forced for all $\vec{\alpha} \in I^{n+1}$).

LEMMA 4.3: *The collection $\{p_{\vec{\alpha}}\colon \vec{\alpha} \in \Pi_{i=0}^{n} H_i\}$ is pairwise compatible.*

Proof: Given $\vec{\alpha}, \vec{\beta} \in \Pi_{i=0}^{n} H_i$ (by the way we picked the pairs $\{H_i\colon i \in \omega\}$), there is always a third increasing sequence $\vec{\gamma} \in I^{n+1}$, such that the relative positions of the elements of $\vec{\alpha}$ with respect to the elements of $\vec{\beta}$ (in their natural order as indiscernibles) is the same as the relative positions of the elements of $\vec{\alpha}$ with respect to the elements of $\vec{\gamma}$ and of $\vec{\beta}$ with respect to the elements of $\vec{\gamma}$.

To show this, let $\vec{\alpha}, \vec{\beta} \in \Pi_{i=0}^{n} H_i$, $\vec{\alpha} = \{a_0, \ldots, a_n\}$ and $\vec{\beta} = \{b_0, \ldots, b_n\}$, both in increasing order.

For each $i = 0, 1, \ldots, n$, we define $c_i \in I$ as follows:

if $a_i = b_i$, then $c_i = a_i$,

if $a_i < b_i$, then pick c_i such that $a_i < b_i < c_i < \min H_{n+1}$,

if $b_i < a_i$, then pick c_i such that $\max H_{n+1} < c_i < b_i < a_i$,

always making sure to pick $c_i < c_{i+1}$.

Notice that by indiscernibility, all the conditions in $\{p_{\vec{\alpha}}\colon \vec{\alpha} \in I^n\}$ have the same type, i.e. their domains all have the same countable order type and they take exactly the same values in corresponding elements of their domains.

For example, if the formula "ξ is the η-th element of the domain of $p_{\vec{\alpha}}$ and $p_{\vec{\alpha}}(\xi) = 0$" is satisfied (in the structure V_κ) by some increasing $\vec{\alpha} \in I^n$ for some ordinal $\xi \in \mathrm{dom}(p_{\vec{\alpha}})$, then it is satisfied by any other increasing sequence of indiscernibles of length n, since $p_{\vec{\alpha}}$, order type of domain of $p_{\vec{\alpha}}$, η-th element of domain of $p_{\vec{\alpha}}$ are all definable in the structure $\langle V_\kappa, \mathbb{P}_1, <, \Vdash, \xi\rangle_{\xi\in\omega_1}$.

If $p_{\vec{\alpha}} \perp p_{\vec{\beta}}$, then there is a least $\xi \in \mathrm{dom}(p_{\vec{\alpha}}) \cap \mathrm{dom}(p_{\vec{\beta}})$ on which they differ. There are thus ordinals $\eta \neq \nu \in \omega_1$ such that ξ is the η-th element of the domain of $p_{\vec{\alpha}}$ and ξ is the ν-th element of the domain of $p_{\vec{\beta}}$. Let γ be as above, then the sequences of indiscernibles $\vec{\alpha} \cup \vec{\beta}$, $\vec{\alpha} \cup \vec{\gamma}$ and $\vec{\beta} \cup \vec{\gamma}$ satisfy the same formulas in $\langle V_\kappa, \mathbb{P}_1, <, \Vdash, \dot{f}, \xi\rangle_{\xi\in\omega_1}$. Therefore,

η^{th} element of $\mathrm{dom}(p_\alpha) = \nu^{th}$ element of $\mathrm{dom}(p_\beta)$,

η^{th} element of $\mathrm{dom}(p_\alpha) = \nu^{th}$ element of $\mathrm{dom}(p_\gamma)$, and

η^{th} element of $\mathrm{dom}(p_\beta) = \nu^{th}$ element of $\mathrm{dom}(p_\gamma)$.

This is a contradiction since it implies that ν^{th} element of $\mathrm{dom}(p_\gamma)$ is two different things at the same time. ∎

Remark: Somewhat more general results than 4.3 are possible; not restricting ourselves to sequences in the products $\Pi_{i=0}^{n} H_i$ but instead allowing finite sequences formed by at most one element in each of the pairs H_i, permits getting a stronger form of homogeneity.

Put now, for each $n \in \omega$,

$$p_n = \bigcup\{p_{\vec{\alpha}}\colon \vec{\alpha} \in \Pi_{n=0}^{n-1} H_i\}.$$

We obtain thus a decreasing sequence $p_1 \geq p_2 \geq \cdots$. The condition $p_\infty = \bigcup_{n\in\omega} p_n$ forces $\dot{f}(\vec{\alpha}) = i_n$ for each $n \in \omega$ and each $\vec{\alpha} \in \Pi_{n=0}^{n} H_i$, so p_∞ forces that $\dot{f}$ is not a counterexample to $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.

To finish the proof of the theorem we have to show how to arrange that in the forcing extension κ is the least cardinal satisfying $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$. We use the following result of Baumgartner, and Devlin and Paris.

LEMMA 4.4 (see [DP, Theorem 2]): *If $V = L$, then for every cardinal κ,* $\mathrm{Fr}(\kappa,\omega)$ *if and only if $\kappa \to (\omega)^{<\omega}$.*

This together with Lemma 2.3 leads to the following interesting fact.

COROLLARY 4.5: *If $V = L$, then for every cardinal κ, $\kappa \to (\omega)^{<\omega}$ if and only if* $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.

In [Si], Silver proved that if $\kappa \to (\omega)^{<\omega}$, then the same holds in L. Using the same argument one can show the following:

LEMMA 4.6: *If $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$ holds, then $L \models \kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.*

Theorem 4.4 together with Lemmas 2.3 and 4.6 is what we need to complete the proof Theorem 4.1.

Let κ be the first ordinal satisfying $\kappa \to (\omega)^{<\omega}$in L. Forcing over L, as above, we obtain a generic extension $L[G]$ where $\kappa = 2^{\aleph_1}$ and $\kappa \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{<\omega}$.

If in $L[G]$ there is a cardinal $\lambda < \kappa$ satisfying the partition property, then by Lemma 4.6, λ also satisfies this property in L, and therefore, by Corollary 4.5, it satisfies $\lambda \to (\omega)^{<\omega}$ in L, contradicting the choice of κ as the minimal ordinal satisfying this partition relation in L. ∎

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